An Interacting Agents Model Approach
To Stock Market Crashes

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Abstract

The aim of this study is to understand the role of herd behaviour on stock market crashes. A model of interacting agents in a market, buying and selling a single financial asset is studied. Agents give their decisions to buy or sell according to a combination of neighbourhood influence, public news and personal judgements. Assuming public news gets worse progressively, the evolution of the agents’ average opinion (based on their decisions) is investigated in the presence of weak and strong neighbourhood influence. Model also suggests a criteria for the existance of herd behaviour under such an assumption. Applying this criteria to the empirical data, evidence of herd behaviour is found in US stock market crash during 2008 financial crisis.

I. Introduction

Traditional economic theory is based on a representative agent model in which an individual behaves in order to maximize his utility and this behaviour produces the main characteristics of aggregate variables of the market (Cont (1999)). This approach has been severely critisized (Kirman, 1992) as it ignores two important facts of real markets: interaction between market participants and differences in their behavioral beliefs and intentions (heterogeneity). Even in early times, Keynes (1936) mentioned of individual physiolog and speculation as important factors for asset prices. A similar approach came from Nobel prize winner Simon (1957) later.

In theory of complex systems with interacting units, it is well known that aggregate behaviour of a system usually arise from the interactions among its units, not from complexity of extraneous factors or that of the units themselves (Mantegna and Stanley (2000), Voit (2003)) and this interaction can bring out aggregate behaviour which is very different from individual one. Many financial economists have realized the importance of this concept therefore modeling economic and financial systems with interacting agents has become very popular lately.

II. Literature

In financial literature, herding is usually defined as individuals imitating each other in their decisions which eventually causes inefficient outcomes for the market as a whole (Bikhchandi and others, 1992). The concept of herding was already a topic of interest in physiolog before considered in financial literature; for example Asch (1952) studied the effect of social environment on individual’s decision and concluded that most of the time individuals neglect their own ideas and agree with the group’s average opinion. Later, models including herding were introduced to the

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finance literature by Shiller (1987), Banerjee (1992) and Bikhchandani and others (1992); they discussed possible effects of herding on overall functioning of financial markets. Scharfstein and Stein (1990) discussed evidence of herding in the behaviour of fund managers, a similar study came from Grinblatt and others (1995) and Wermers (2000). Lakonishok and others (1992) reported that pension fund managers trade the same stocks that others’ trade and this creates a positive feedback strategy. Trueman (1994) and Welch (2000) studied the effects of herding on use of valuable information and highlighted that herding leads to inefficient decision making. Olsen (1996) showed that, between 1985-1987, analysts herded in %52 to %72 of their submitted forecasts. De Bondt and Forbes (1999) confirmed these results by investigating herd behaviour in UK data between 1986-1997 (See also Sias (2004)). For an overview of theories on herding in capital markets see Hirshleifer and Teoh (2003).

There has been several attempts to model the stock market dynamics in history. In the early years, usually linear models were used but the problem is; in these models when external random shocks do not exist, stock price fluctuations would be absent; furthermore statistical analysis revealed strong nonlinearities in the empirical data. In this situation, its ability to generate large movements which appear to be random with greater frequency than linear models made nonlinear dynamics and chaos theory very attractive. Many researchs have been conducted on financial markets by using nonlinearity and chaos. Grandmont (1987, 1992) tried to find whether simple nonlinear deterministic models can demonstrate the kind of fluctuations usually found in financial data and argued that it is important incorporate nonlinearities to study such fluctuations. Some other works discussed the possibility that if empirical financial time series could be characterized by chaotic dynamics. (For a collection of these works see LeBaron (1991)). While the early studies on nonlinear dynamics and chaos by Scheinkman and Lebaron (1989) suggested a possibility of low dimensional chaos in the stock markets and economical data (Hommes 1991), later researchs by Hsieh (1991), Liu and others (1992), Abhyankar and others (1995, 1997), Cecen Erkal (1996) and Pandey and others (1998) found little evidence supporting chaotic dynamics in such data.

In recent years several interacting agent based models have been developed to understand the dynamics of financial markets. Bouchaud and Cont (2000) used noise traders who are subject to herding. Similar approach came from Cincotti and others (2001). Lux and Marchesi (2000) studied a model in which two types of traders, namely “fundamentalists” and “chartists”, interact. Models including order books were also introduced (Bak and others (1997), Chiarella and Iori (2002) and Maslov(2000)). Other popular models were studied by Aoki (1997), Bak and others (1993), Bak and others (1998), Caldearel and others (1997), Cont (1998), Herreiner and others (1995), Kirman (1994), Levy and others (1995), Lux (1998) and Penna and Stauffer (1998).

In constructing these interactive models, different kinds of popular models in fundamental sciences such as coupled map lattice models, sandpile models, noise trader models and etc. have been widely used. In this study, we will use the Ising model which is a model of ferromagnetism.

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1 For an introduction to the subject see Kelsey (1988), Baumol and Behabib (1989) or Scheinkman and LeBaron(1990)
One of the very first attempts to explicitly model the interaction between market participants was made by Föllmer (1974). He used theory of the Ising model to construct an analogy between market participants and interacting particles and showed that in a market where people give binary decisions based on each other, multiple equilibria could exist and these conditional decisions do not necessarily determine the market equilibrium in a unique way. Later Brock (1995) used Ising model to understand the asset price behaviour, and Brock and Hommes (1995) used it to analyse expectation formation in the market. Similar approaches came from Durlauf (1996a,b and 1997) and Brock and Durlauf (1996) to analyze economical and social decision making. These studies argued that the knowledge of private individual preferences is not sufficient to predict the collective behaviour; as pointed out by Durlauf (1997): “When individual behaviour is driven by a desire to be similar to others, this does not provide any information on what they actually do; rather it merely implies that whatever behaviour occurs, there will be substantial within-group correlation due to the conformity effects”.

A reason why Ising model is of interest is that, it is a system made up of many subunits. The subunits in an Ising model are the interacting spins, and the subunits in the economy are market participants-buyers and sellers. During any time interval, these subunits of the economy may be either positive or negative as regards perceived market opportunities. People interact with each other and this fact produces what economists call the herd effect. The orientation of whether they buy or sell is influenced not only by neighbours but also by news usually realized by a global external field. If we hear bad news, we may be tempted to sell. In addition, we may naturally assume that the habit of individual’s own also plays an important role in such a decision so the decision of any agent may be written as a function of the other agents’ decisions and parameters for public news (externality) and idiosyncrasy.

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2 Coupled map lattices are discrete in time and space but have a continuous state space. These are simple models that are capable of displaying complex dynamical behaviour generic to many spatiotemporal systems (Kaneko (1989, 1993)).

3 This model is an example of a dynamical system displaying self organized criticality in physics. It is a cellular automaton. At each site on the lattice, there is a value that corresponds to the slope of the pile. This slope builds up as grains of sand are randomly placed onto the pile, until the slope exceeds specific threshold value at which time that site collapses transferring sand into the adjacent sites, increasing their slope. Thus random placement of sand at a particular site may have no effect or it may cause a cascading reaction that will affect every site on the lattice.

4 In finance, noise traders usually refer to irrational traders. De Long and others (1990) and Cutler and others (1989) argue that large price movements in the market develop from noise traders. De Long and others (1990) also suggest that noise trader behaviour drives security prices from their underlying fundamental values. Furthermore, since arbitrage has practical limits, rational investors fail to fully offset the effects of noise trader behaviour so “noise trader risk” becomes a priced factor in their model which theoretically enabling noise traders to earn larger returns than rational investors.

5 Shiller on the U.S. stock market crash of 1987 and interacting agent models: “The suggestion we get of the causes of the crash is one of people reacting to each other with heightened attention and emotion, trying to fathom what other investors were likely to do and falling back on intuitive models like models of price reversal or continuation. There appears to be no recognizable exogeneous trigger for the crash. With such popular models, a feedback system is created with possibly complicated dynamics and we do not need to refer to a trigger to explain a crash (Shiller, 1990)’’

6 News media content has been shown to be an important factor shaping investor sentiment. (Tetlock, 2007 and (Macskassy and others, 2008))

By following the approach mentioned above, to understand the role of herd behaviour on stock market crashes, a model inspired by the work of Dahmen and Sethna (1996) will be considered (similar idea comes from Bouchaud and Michard (2005) to study the herd effect on birth rates and cell phone sales in Europe). Then the theoretical results of this model will be applied to empirical data to see if herd behaviour was effective in the drastic drops of major U.S. market indices during 2008 financial crisis.

III. Model

Consider a network of agents buying and selling a single financial asset. Each agent is indexed by an integer \( i = 1, 2, ..., N \) and \( N \) is assumed to be very large. The agents who are directly connected to agent \( i \) are called neighbours of \( i \). We assume that agent \( i \) can have two decisions: \( \varphi_i \in \{+1, -1\} \) where +1 and −1 refers to buy and sell respectively. The decision of agent \( i \) is determined by;

\[
\varphi_i = \text{sign} \left( \sum_j I_{ij} \varphi_j + P + \epsilon_i \right) \quad (E1)^8
\]

where \( I_{ij} = 1/d \) (\( d \) is a positive constant and \( d \) is the number of neighbours of agent \( i \)), denotes the tendency towards imitation. It shows how strongly agent \( j \) has influence on agent \( i \) and the sum runs over neighbours \( j \) of \( i \) (i.e. neighbours have direct effect on an agent’s decisions).

\( P \in (-\infty, \infty) \) is the term for public news which has a global influence on all agents. For example; if \( P > 0 \) (denoting good news), agents will tend to buy.

\( \epsilon_i \) is the personal judgement of agent \( i \). We may assume a normal distribution (due to the central limit theorem) for \( \epsilon_i \) with mean zero and variance \( \sigma^2 \),

\[
\rho(\epsilon_i) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \frac{\epsilon_i^2}{\sigma^2}}
\]

(\( \sigma \) can be considered as a heterogeneity parameter. For larger \( \sigma \), agents’ decisions become more irregular and unpredictable)

Note that \((E1)\) describes the decision of an agent at a given point in time. In the next instant, new \( \epsilon_i \)'s are drawn, neighbourhood interaction creates new influences on agents, and agents may change their decisions. Based on these decisions, agents’ average opinion \( A \) about the market at a specific time \( t \) is formed in the following way;

\[
A(t) \equiv \frac{\sum_i \varphi_i(t)}{N}
\]

One approach to solve this model is to replace the interaction of an agent with his/her neighbours by an interaction with the average opinion in the market. This is

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8 This is analogous to random field Ising model in physics
mean-field\(^9\) approximation where every agent is influenced by remaining others with equal strength (in real life, there is no natural topology for the interaction between agents, markets are assumed to have large number of agents and each agent has possibly many neighbours so a mean field approach seems appropriate). In this case, new neighbourhood influence is of size \(I_{ij} = I/N\) (every agent is a neighbour of \(i\) now) then new decision criteria is the following;

\[
\varphi_i = \text{sign} \left( IA + P + \varepsilon_i \right)
\]  

(E2)

We start with a situation where \(P\) is very large i.e. public news is all good about the market hence everyone is optimistic and all agents have \(a\ priori\) decision to buy. Now suppose things are getting worse and \(P\) is decreasing progressively.

Every agent \(i\) changes his/her decision when \(IA + P + \varepsilon_i\) changes sign. At any \(P\) level, all agents with \(IA + P + \varepsilon_i > 0\) will still give a decision to buy and agents with \(IA + P + \varepsilon_i < 0\) will give a decision to sell.

The probability that the personal judgement of agent \(i\) lies between \(\varepsilon_i\) and \(\varepsilon_i + d\varepsilon_i\) is \(\rho(\varepsilon_i)d\varepsilon_i\), then we have the following self consistency relation

\[
A = 1 - 2 \int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i = 1 - 2F(-IA - P)
\]  

(E3)

where \(F\) is cumulative distribution function of \(\rho\).

Observe that if there wasn’t any interaction among agents, by (E3) we would have,

\[
A = 1 - 2 \int_{-\infty}^{-P} \rho(\varepsilon_i)d\varepsilon_i = 1 - 2F(-P)
\]  

(E4)

hence average opinion would be decreasing smoothly for any \(P\). (see fig. 1.a)

If neighbourhood influence is weak enough (i.e. for small enough \(I\)), expanding R.H.S. of (E3) in powers of \(I\) leads to;

\[
A \approx \frac{1 - 2F(-P)}{1 - 2I\rho(P)}
\]  

(E5)

so, in opposition to no interaction case, as \(\rho(P)\) gets close to its maximum value, neighbourhood influence \(I\) leads to overreaction, and as it gets stronger and exceeds a critical value \(I_c\);

\[
I_c = \sqrt{\frac{\pi}{2}} \sigma
\]  

(E6)

\(^9\) Mean field methods were originally introduced and developed for calculations of the spin-glass models in quantum mechanics. Similar approach is also used often in artificial neural networks due to the isomorphism between Hopfield model and zero temperature Ising model (see Amit (1989) for details). For more on mean field theory see Uzunov (1992).
slope of average opinion diverges when $P$ reaches a critical level $P_c (I)$, i.e. if neighbourhood influence is strong enough ($I$ is sufficiently large), as $P$ is decreased progressively, average opinion (which is positive at that moment) will first decrease smoothly then at a point, it will jump down to a negative value (happens when $P$ reaches its critical level $P_c (I)$ for given $I$), then keep on decreasing (see fig. 1.b).

Thus, in the presence of strong enough neighbourhood influence, as public news gets worse, considerable amount of agents suddenly change their decisions from buy to sell whereas this transition would be smooth if neighbourhood influence was weak or did not exist at all.

Furthermore, an analogous result of Dahmen and Sethna (1996) shows that when neighbourhood influence is close to its critical value, there is a universal scaling law taking the form:

$$\frac{dA}{dP} = \frac{1}{(I_c - I)} G \left( \frac{P - P_c}{(I_c - I)^{3/2}} \right)$$  \hfill (E7)

Figure 1: Evolution of Average Opinion under Weak and Strong Neighbourhood Influence

Where $G(y = 0)$ is constant and $G(y \rightarrow \infty) \sim y^{-2/3}$, implying that if we approximate $G$ with a Gaussian function then; when neighbourhood influence is strong enough, the peak of the slope of the average opinion has an height $h$ and width $w$ related by the scaling law of the form $h \sim w^{-2/3}$ whereas in case of no interaction among agents, this would be $h \sim w^{-1}$ (see appendix 3 for a clear picture).

3.1 Methodology

Weekly values of six major US market indices; DJI, S&P 500, NASDAQ 100, NYSE US100, Amex Composite and Russell 3000, is considered in this work. For a given time interval, changes in these indices can be interpreted as a change in average opinion of participants in US stock market.
The timeline chosen for the analysis is the 75 weeks between October 2007 and March 2009 (to be specific: for DJI, S&P 500 and NASDAQ 100, each Friday’s closing value from 5.10.2007 to 6.3.2009 and for NYSE US100, Amex Composite and Russell 3000, each Monday’s closing value from 1.10.2007 to 2.3.2009 were taken). The reason behind this choice is, it is approximately the duration in which rumours got really serious about housing market having big problems, and each week after, more and more bad news about finance sector came from global media. This interval stands for the time when public news is getting worse progressively.

The index data is rescaled such that during this 75 weeks, maximum and minimum value for each index is +1 and −1 respectively.

The slope of average opinion is then taken as the weekly changes in rescaled indices: to mathematically formalize the “public news getting worse progressively”, we take \( P \propto t \) where \( t \) denotes time (\( t = 0 \) for the initial week and increases by one for each week after that) hence “\( P \) decreases with a constant rate over time”\(^{11}\) and consequently we have

\[
dA/dP \propto dA/dt = A_t - A_{t-1}
\]

\( (E8) \)

Data for the slope obtained by \( (E8) \) will be noisy, a suitable choice to capture the height \( h \) and width \( w \) of the peak is to use a Gaussian fit for the slope of the average opinion i.e.

\[
A_t - A_{t-1} \approx c + h e^{-\frac{(t-t')^2}{2w^2}}
\]

\( (E9) \)

For each index, \( h \) and \( w \) is estimated from \( (E9) \) by a simple non-linear OLS regression (observation: \( c \ll h \)). Using these \( h \) and \( w \) values, the relation \( h = aw^{-\theta} \) is investigated by a simple log-linear regression to see if neighbourhood influence was strong during this time interval.

3.2 Results

For each index, evolutions of average opinion and slope are given in figure A.1, A.2, A.3, A.4, A.5, A.6 in the appendix 1. The Gaussian fit for the slopes of the indices gives us the following heights and widths for the peaks;

<table>
<thead>
<tr>
<th></th>
<th>DJI</th>
<th>S&amp;P 500</th>
<th>NASDAQ</th>
<th>NYSE US</th>
<th>Amex</th>
<th>Russell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height</td>
<td>0.482616</td>
<td>0.43584</td>
<td>0.360427</td>
<td>0.453288</td>
<td>0.68801</td>
<td>0.436874</td>
</tr>
<tr>
<td>Width</td>
<td>0.440129</td>
<td>0.512292</td>
<td>0.890226</td>
<td>0.453609</td>
<td>0.361616</td>
<td>0.549284</td>
</tr>
</tbody>
</table>

\(^{10}\) For a nice justification of this assumption see Kolb (2011) or Ozatay (2009).

\(^{11}\) Having a constant \( dP/dt \) is necessary: \( (E7) \) holds with \( P \) as the independent variable but rescaling to the form \( (E9) \) with \( t \) as the variable also rescales \( h \) and \( w \) by \( dP/dt \).
and the log-linear relationship between them is as the following;

**Figure 2: Log-linear relationship between \( h \) and \( w \)**

![Log-linear relationship between \( h \) and \( w \)](image)

The best fitted line gives us \( \theta = -0.605308 \)

Considering the small sample size and a standard error of 0.173381, we can reject \( \theta = -1 \) while we can not reject \( \theta = -2/3 \).

**IV. Conclusion and Discussion**

An Ising model of interacting agents in a market, buying and selling a single financial asset is studied in this work. In this model, agents give their decisions to buy or sell according to a combination of neighbourhood influence, public news and personal judgements. Based on these decisions, agents’ average opinion about the market is formed. It is found that for small neighbourhood influence or strong diversities in agents’ judgements, average opinion decreases continuously as public news get worse. In the same scenario, if neighbourhood influence is strong enough, a jump occurs in average opinion around a critical time when a group of agents suddenly change their decisions from buy to sell which causes a crash. Around this time, the peak of the slope of the average opinion, has an height \( h \) and width \( w \) related by \( h \sim w^{-2/3} \) whereas in the case of no interaction among agents, this relation would be \( h \sim w^{-1} \).

These results were applied to the empirical data by considering weekly values of six major US market indices during the time interval between October 2007 and March 2009. Evidence of herd behaviour is found in the drastic drops of these indices.\(^{12}\)

If we want to get an answer in an analytical way then mean field approximation is the way to go. Basically, replacing the decision criteria \((E1)\) by \((E2)\) takes into account the heterogeneity of agents and the influence of \( \phi_j \) on \( \phi_i \) but not the fact that this \( \phi_i \) again influences \( \phi_j \) creating a feedback so mean field approximation is more accurate if agents have large number of neighbours. To see this fact, we have the

\(^{12}\) For a nice point of view supporting this result see Soros (2009)
following simulation: Every agent is placed on a point in a cubic integer lattice in 3 dimensions hence each agent has $d = 6$ neighbours (periodic boundary conditions are applied for the agents on the boundaries of the cube) and we have $N = 60^3$ agents in total. The simulation tests the effects of different $\sigma$ values when interaction effect is taken as $I = 1$. As in (E6); mean field theory states that for given $I = 1$, critical $\sigma$ would be $\sqrt{\frac{2}{\pi}} I \approx 0.798$, but test suggests that if each agent has $d = 6$ number of neighbours, this critical $\sigma$ value would be somewhere between 2.1 and 2.2 which definitely shows mean field approach undershoots the true value here: as mentioned before this approach is more accurate when number of neighbours gets larger.

Although this model has strong assumptions, it may at least help us to understand the importance of interaction and heterogeneity of market participants in stock market crashes. For future work, a few improvements in the model is considered: Interaction among agents can be time dependent; evolving subject to some criteria, and agents can have biased personal judgements characterizing them as optimists and pessimists.
References


Kaneko, K., Physica D 34: 1, 1989


Appendix 1

Fig. A.1: Evolution of average opinion and its slope in DJI

Fig. A.2: Evolution of average opinion and its slope in S&P 500

Fig. A.3: Evolution of average opinion and its slope in Russell 3000
Fig. A.4: Evolution of average opinion and its slope in NYSE US100

Fig. A.5: Evolution of average opinion and its slope in Amex Composite

Fig. A.6: Evolution of average opinion and its slope in NASDAQ100

Red arrows in figures A.1 through A.6 show the event of a group of agents suddenly changing their decisions which causes a crash and during this time interval, divergence of the slopes can be seen by red points on the RHS.
Appendix 2

**E3:** At any $P$ level, all agents $i$ with $\varepsilon_i > -IA - P$ will still give a decision to buy; the probability of this is,

$$\int_{-IA-P}^{\infty} \rho(\varepsilon_i)d\varepsilon_i$$

similarly, agents $i$ with $\varepsilon_i < -IA - P$ will give a decision to sell, which has the probability

$$\int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i$$

then the average opinion is,

$$A = -1\{ \int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i \} + 1\{ \int_{-IA-P}^{\infty} \rho(\varepsilon_i)d\varepsilon_i \}$$

$$= -\int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i + [1 - \int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i] = 1 - 2\int_{-\infty}^{-IA-P} \rho(\varepsilon_i)d\varepsilon_i = 1 - 2F(-IA - P)$$

**E4:** clearly follows from *(E3)*

**E5:**

$$A = 1 - 2F(-IA - P)$$

$$= 1 - 2\{F(-P) + (-IA)F'(-P) + \cdots \}$$

$$\approx 1 - 2\{F(-P) - IA\rho(-P)\}$$

$$= 1 - 2F(-P) + 2IA\rho(P)$$

$$\Rightarrow A(1 - 2I\rho(P)) \approx 1 - 2F(-P)$$

$$\Rightarrow A \approx \frac{1 - 2F(-P)}{1 - 2I\rho(P)}$$
E6: Remember (E3),
\[ A(P) = 1 - 2 \int_{-\infty}^{-1A-P} \rho(\varepsilon_i) d\varepsilon_i \]  
(E3)

\( \rho \) is symmetric distribution hence for (E3), \( A(0) = 0 \) is the trivial solution at \( P = 0 \). If \( A(0) = 0 \) is the only solution at \( P = 0 \) then there is no jump in the average opinion. To have non-trivial solutions for \( A(0) \), the slope of the R.H.S. of (E3) (as a function of \( A(0) \)) must be larger than 1 at \( A(0) = 0 \).

At \( P = 0 \) and near \( A(0) = 0 \), we can approximate R.H.S. of (E3) in the following way;

\[
1 - 2F(-IA) \approx 1 - 2[F(0) + (-IA(0))F'(0) + \cdots]
= 1 - 2\left[ \frac{1}{2} + (-IA(0))\rho(0) + \cdots \right]
\approx 2I\rho(0)A(0)
\]

then (E3) has multivalued solution if we have

\[ 2I\rho(0) \geq 1 \]

then combining with the fact

\[ \rho(0) = \frac{1}{\sigma\sqrt{2\pi}} \]

we have

\[ I_c = \frac{\sqrt{\pi}}{\sqrt{2}} \sigma \]

which is not surprising since as agents heterogeneity gets stronger, in order to behave as a collective group having the same decision, they should have stronger influence on each other.

E7: An outline of derivation of the (E7) is the following (Dahmen and Sethna (1996)); Observe that;

\[ A = 1 - 2 \int_{-\infty}^{-1A-P} \rho(\varepsilon_i) d\varepsilon_i = 1 - 2F(-IA - P) \]

\[
\frac{dA}{dP} = -2[\rho(-IA - P) \frac{d}{dP}(-IA - P)]
= 2\rho(-IA - P) \left( I \frac{dA}{dP} + 1 \right)
\]
\[
\frac{dA}{dP} = 2\rho(-IA - P) + \frac{dA}{dP} [21\rho(-IA - P)]
\]

\[
\Rightarrow \frac{dA}{dP} = \frac{2\rho(x)}{1 - 21\rho(x)} \quad \text{where} \quad x = -IA - P
\]  

(1)

hence slope of the average opinion diverges if

\[
1 - 21\rho(x_c) = 0
\]  

(2)

which also defines \(x_c \equiv -IA(P_c) - P_c\).

Expanding around such a critical point gives;

\[
1 - 21\rho(x) = 1 - 21[\rho(x_c) + \rho'(x_c)(x - x_c) + \frac{1}{2}\rho''(x_c)(x - x_c)^2 + \cdots]
\]

\[
((2) \Rightarrow) = -21[\rho'(x_c)(x - x_c) + \frac{1}{2}\rho''(x_c)(x - x_c)^2 + \cdots]
\]

then combining with (1) we have

\[
\frac{dA}{dP} = \frac{\rho(x_c)}{-1[\rho'(x_c)(x - x_c) + \frac{1}{2}\rho''(x_c)(x - x_c)^2 + \cdots]}
\]  

(3)

\(\rho\) is normal distribution, hence it is analytic with one maximum and \(\rho'' \neq 0\) so (3) gives us two cases to consider;

\[\rho'(x_c) = 0 \text{ and } \rho'(x_c) \neq 0\]

For \(\rho'(x_c) = 0\); it follows that \(x_c = 0\) hence \(\rho(x_c) = 0 = \frac{1}{\sigma\sqrt{2\pi}}\) and also by (2) we have \(\rho(x_c) = \frac{1}{21}\) implying

\[
I = \frac{\pi}{\sqrt{2}} \sigma \quad (= I_c)
\]

so \(\rho'(x_c) = 0\) occurs when “\(I\) is close to its critical value” (the case \(\rho'(x_c) \neq 0\) is found for \(I > I_c\)) and in this situation integrating (3) leads to the following leading order scaling behaviour;
\[ A(P) - A(P_c(I_c)) \sim (I_c - I)\beta \Lambda \left( \frac{P - P_c}{(I_c - I)^\beta \delta} \right) \] (4)

for small \((P - P_c)\) and \((I_c - I)\). \(\Lambda\) is the universal scaling function and the mean field exponents are \(\beta = \frac{1}{2}\) and \(\beta \delta = \frac{3}{2}\).

To get rid of \(A(P_c(I_c))\) we take derivative of the average opinion with respect to \(P\):

\[
\frac{dA}{dP} \sim (I_c - I)^{\beta - \beta \delta} \hat{\Lambda} \left( \frac{P - P_c}{(I_c - I)^{\beta \delta}} \right)
\]

\[
= \frac{1}{(I_c - I)} \hat{\Lambda} \left( \frac{P - P_c}{(I_c - I)^{3/2}} \right)
\]

where \(\hat{\Lambda}\) is the derivative of \(\Lambda\) with respect to its argument \(\frac{(P - P_c)}{(I_c - I)^{\beta \delta}}\).
Appendix 3

A Gaussian function is a characteristic symmetric “bell curve” shaped function that quickly falls of towards $+/-\infty$. It is a function of the form

$$f(x) = c + h e^{-\frac{1}{2} \frac{(x-y)^2}{w^2}}$$ \hspace{1cm} (E10)

The parameter $|h|$ is the height of the curve’s peak, $\gamma$ is the position of the center of peak, $w$ controls the width of the bell and $c$ sets the distance from horizontal axis.

Remember, when neighbourhood interaction is strong enough, we have the following relation as in (E7),

$$\frac{dA}{DP} = \frac{1}{(I_c-I)} G \left( \frac{P-P_c}{(I_c-I)^{3/2}} \right)$$

In this case approximating $G$ with a Gaussian form as in (E10) relates $h$ and $w$ by $h \sim (I_c-I)^{-1}$ and $w \sim (I_c-I)^{3/2}$ hence the term $h$ behaves like $h \sim w^{-2/3}$.

On the other hand, in case of no interaction as in (E4), we have

$$A = 1 - 2F(-P) \Rightarrow \frac{dA}{DP} = 2\rho(-P) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} e^{-\frac{1}{2} \frac{p^2}{\sigma^2}}$$

In this case the slope of average opinion ($dA/DP$) is already in Gaussian form and comparing this fact with (E10) relates $h \sim \sigma^{-1}$ and $w \sim \sigma$ hence we have $h \sim w^{-1}$.

In the model studied in this work, if necessary time is given, the decreasing $P$ (which is an exogeneous factor) will enforce the system to be in an average state close to $-1$ whether neighbourhood interaction exists or not as it can be clearly understood from (E1) or (E2). But in the given scenario (decreasing $P$), strong enough interaction among the agents (which is an endogeneous factor) will serve as an accelerator in drops of the average opinion which also creates the affect of faster decay in the average opinion’s slope just before this slope reaches it’s negative peak level. So approximating $G$ with a Gaussian form allows us to compare the heights and widths of the bells in both cases. The reason why we approximate $G$ with a Gaussian is because the theory suggests it as it can be understood when we think how the average opinion’s slope in figure 1.b changes in time: starting from a negative value close to zero, the slope of the average opinion will decrease very slowly, then in a short time interval it will reach to its negative peak level, then again it will take a negative value close to zero and will start to increase very slowly approaching to zero which can be characterized by a function of the form (E10) with a negative $h$. 